

Prove that $\sum_{i=1}^n [(i-1) \cdot (i-1)! + 1] = n! + n - 1$ for all positive integers n using mathematical induction. SCORE: ____ / 15 PTS

Basis case: $\sum_{i=1}^1 [(i-1) \cdot (i-1)! + 1] = 0 \cdot 0! + 1 = 1 = 1! + 1 - 1$

GRADED BY
ME

Inductive step: Assume that $\sum_{i=1}^k [(i-1) \cdot (i-1)! + 1] = k! + k - 1$ for some arbitrary integer $k \geq 1$

Prove that $\sum_{i=1}^{k+1} [(i-1) \cdot (i-1)! + 1] = (k+1)! + k + 1 - 1 = (k+1)! + k$

$$\begin{aligned} & \sum_{i=1}^{k+1} [(i-1) \cdot (i-1)! + 1] \\ &= \sum_{i=1}^k [(i-1) \cdot (i-1)! + 1] + k \cdot k! + 1 \\ &= k! + k - 1 + k \cdot k! + 1 \\ &= k! + k \cdot k! + k \\ &= (1+k) \cdot k! + k \\ &= (k+1)! + k \end{aligned}$$

So, by mathematical induction, $\sum_{i=1}^n [(i-1) \cdot (i-1)! + 1] = n! + n - 1$ for all positive integers n

Consider the expansion of $(3x^{10} + \frac{5}{x^2})^{24}$.

SCORE: ____ / 9 PTS

For the questions below, you may write the coefficients in your final answers in factored form, as shown in lecture.

The coefficients in your final answers must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

NOTE: Do NOT use your calculator's ! nor $C(n, r)$ features.

$\left(\frac{1}{2}\right)$ POINT EACH EXCEPT AS NOTED

[a] Find the fifth term in the expansion.

$$\binom{24}{4} (3x^{10})^{24-4} \left(\frac{5}{x^2}\right)^4 = \frac{24!}{4!20!} 3^{20} x^{200} 5^4 x^{-8} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 20!} 3^{20} 5^4 x^{192} = 23 \cdot 22 \cdot 21 \cdot 3^{20} 5^4 x^{192}$$

[b] Find the coefficient of x^{12} in the expansion.

$$\sum_{r=0}^{24} \binom{24}{r} (3x^{10})^{24-r} \left(\frac{5}{x^2}\right)^r = \sum_{r=0}^{24} \binom{24}{r} 3^{24-r} x^{10(24-r)} 5^r x^{-2r} = \sum_{r=0}^{24} \binom{24}{r} 3^{24-r} 5^r x^{240-12r}$$

$$x^{240-12r} = x^{12} \Rightarrow 240 - 12r = 12 \Rightarrow 20 - r = 1 \Rightarrow r = 19$$

$$\binom{24}{19} 3^{24-19} 5^{19} = \binom{24}{19} 3^5 5^{19} = \frac{24!}{19!5!} 3^5 5^{19} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 19!} 3^5 5^{19} = 23 \cdot 22 \cdot 21 \cdot 4 \cdot 3^5 5^{19}$$

[c] Find the coefficient of x^{30} in the expansion.

$$x^{240-12r} = x^{30} \Rightarrow 240 - 12r = 30 \Rightarrow 40 - 2r = 5 \Rightarrow r = \frac{35}{2} \text{ which is not an integer}$$

No x^{30} term, so coefficient = 0

Expand and simplify $(t^2 - 2\sqrt{t})^4$.

SCORE: _____ / 6 PTS

You may write the coefficients in your final answer in factored form, as shown in lecture.

The coefficients in your final answer must **NOT** contain division, ! nor $C(n, r)$ (or equivalent) notation.

NOTE: Do NOT use your calculator's ! nor $C(n, r)$ features.

$$\begin{aligned} & (t^2)^4 + 4(t^2)^3(-2\sqrt{t}) + 6(t^2)^2(-2\sqrt{t})^2 + 4(t^2)(-2\sqrt{t})^3 + (-2\sqrt{t})^4 \\ & = t^8 - 8t^{\frac{13}{2}} + 24t^5 - 32t^{\frac{7}{2}} + 16t^2 \end{aligned}$$